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LIQUID TRAPPING ON CYLINDER EXTRACTION

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It is important to know the thickness of the film of liquid formed on a cylindrical body, for example in depositing insulation on wires and also in the production of glass and synthetic fibers. The theory of [1-3] is restricted to low extraction velocities. The approach considered below is applicable to a very wide velocity range.

1. Consider a cylinder of radius R extracted at a constant velocity U from a sufficiently large volume of liquid (Fig. 1). The thickness of the film remaining on the surface is determined by the interaction between the internal friction, the mass forces, and the surface tension. The effect of these forces on the trapping are determined primarily by the extraction speed and the properties of the medium.

The liquid in the film is simultaneously extracted by the cylinder and flows under gravity back into the bath. Therefore, at the surface of the film there should be a stagnation line, where the flow direction reverses. The stream lines passing through this separate the part of the liquid carried by the cylinder from the rest in the bath. We write the equations of motion for each of these regions and find the condition for linking up the solutions.

2. We set the z axis along the flow parallel to the cylinder axis, while the r axis is perpendicular to it and passes through the stagnation line. The region of entrainment is bounded from below by a plane perpendicular to the axis of the cylinder and passing through the stagnation line, while upwards it passes into the region of constant film thickness $h_0 = \xi_0 - R$. Physical considerations show that the characteristic dimension L of this region considerably exceeds h_0, i.e., $h_0/L = \varepsilon \ll 1$.

We write the Navier–Stokes equations and the boundary conditions for the extraction region:

$$u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right), \tag{2.1}$$

$$u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v}{\partial r^2}-\frac{v}{r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}\right),$$

$$\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}+\frac{v}{r}=0;$$

$$\frac{\partial u}{\partial r}+\frac{\partial v}{\partial z}+\frac{2d\xi}{dz}\left[1-\left(\frac{d\xi}{dz}\right)^2\right]^{-1}\left(2\frac{\partial v}{\partial r}+\frac{v}{r}\right)=0 \quad \text{at} \quad r=\xi, \tag{2.2}$$

$$p-p_0+\sigma\left[1+\left(\frac{d\xi}{dz}\right)^2\right]^{-\frac{1}{2}}\left\{\frac{d^2\xi}{dz^2}\left[1+\left(\frac{d\xi}{dz}\right)^2\right]^{-1}-\frac{1}{\xi}\right\}=$$
$$=2\mu\left[1-\left(\frac{d\xi}{dz}\right)^2\right]^{-1}\left\{\left[1+\left(\frac{d\xi}{dz}\right)^2\right]\frac{\partial v}{\partial r}+\frac{v}{r}\right\} \quad \text{at} \quad r=\xi;$$

$$u=U,\ v=0 \quad \text{at} \quad r=R; \tag{2.3}$$

$$v=u\frac{d\xi}{dz} \quad \text{at} \quad r=\xi, \tag{2.4}$$

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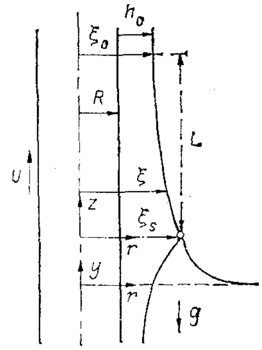

Fig. 1

where z and r are cylindrical coordinates; u and v, velocity components along the z and r axes, respectively; p, pressure; p_0 = const, pressure in the gas; ρ, density; μ and ν, dynamic and kinematic viscosities, respectively; g, acceleration due to gravity; σ, surface tension; and ξ, radius of the free film surface.

Equations (2.2) express the absence of tangential and normal stresses at the surface of the film, while (2.4) is the usual kinematic condition at the free surface. We take the thickness of the extracted film as comparable with the radius of the cylinder R. We introduce the following dimensionless quantities:

$$u^* = u/U,\ v^* = v/\varepsilon U,\ z^* = \varepsilon z/R,\ r^* = r/R,\ p^* = pa/\sigma,$$
$$\mathrm{Ca} = \mu U\ \sigma,\ \mathrm{Go} = R(\rho g/2\sigma)^{1/2},\ \gamma = (\sigma/\rho)(g\nu^4)^{-1/3},\ a = (2\sigma/\rho g)^{1/2}.$$

Here Ca is the dimensionless extraction velocity; Go, Goucher number; a, capillary constant; and γ, parameter relating the physical properties of the liquid. We neglect terms of order ε^2 in (2.1)-(2.4) to get

$$u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right); \tag{2.5}$$

$$\frac{\partial p}{\partial r} = \mu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (rv)}{\partial r}\right],\quad \frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0; \tag{2.6}$$

$$p - p_0 + \sigma\left(\frac{d^2\xi}{dz^2} - \frac{1}{\xi}\right) = 2\frac{\mu}{r}\frac{\partial (rv)}{\partial r} \quad \text{at} \quad r = \xi,\quad \frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = \xi. \tag{2.7}$$

The boundary conditions of (2.3) and (2.4) remain unchanged. The inertial terms in (2.5) are of order $\varepsilon \mathrm{Ca}\mathrm{Go}\gamma^{3/2}$ and may be of order ε^2 or less in accordance with the values of Ca, Go, and γ. We ignore them in what follows.

We integrate (2.6) and use (2.7) to get

$$p - p_0 = \mu\left[\frac{1}{r}\frac{\partial (rv)}{\partial r} + \frac{1}{\xi}\frac{\partial (rv)}{\partial r}\bigg|_{r=\xi}\right] - \sigma\left(\frac{d^2\xi}{dz^2} - \frac{1}{\xi}\right). \tag{2.8}$$

We substitute (2.8) into (2.5) and discard terms of order ε^2 to get the final equations for the extraction region:

$$\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\sigma}{\rho}\left(\frac{d^3\xi}{dz^3} + \frac{1}{\xi^2}\frac{d\xi}{dz}\right) - g = 0; \tag{2.9}$$

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial (rv)}{\partial r} = 0; \tag{2.10}$$

$$u = U \quad \text{at} \quad r = R,\quad \frac{\partial u}{\partial r} = 0 \ \text{at} \ r = \xi,\ v = u\frac{d\xi}{dz} \quad \text{at} \quad r = \xi. \tag{2.11}$$

The equation of continuity (2.10) is represented in integral form, for which purpose we average (2.10) with respect to r from R to ξ and alter the order of differentiation and integration, use the second condition in (2.11), and introduce the liquid flow rate:

$$Q = 2\pi\int_R^\xi ru\,dr,$$

which gives

$$2\pi \frac{\partial}{\partial z} \int_R^{\xi} r u dr = \frac{\partial Q}{\partial z} = 0.$$

Consequently, we have Q = const for the flow rate.

Double integration of (2.9) and satisfaction of the boundary conditions of (2.11) gives

$$u = U - \frac{AR^2}{4\mu} \left(2 \frac{\xi^2}{R^2} \ln \frac{r}{R} - \frac{r^2}{R^2} + 1 \right), \tag{2.12}$$

where

$$A = \rho g - \sigma \left(\frac{d^3 \xi}{dz^3} + \frac{1}{\xi^2} \frac{d\xi}{dz} \right).$$

For Q we get

$$\frac{Q}{2\pi} = \frac{UR^2}{2} \left(\frac{\xi^2}{R^2} - 1 \right) - \frac{AR^4}{4\mu} \left(\frac{\xi^4}{R^4} \ln \frac{\xi}{R} - \frac{3\xi^4}{4R^4} + \frac{\xi^2}{R^2} - \frac{1}{4} \right). \tag{2.13}$$

All the derivatives of the film thickness with respect to z are zero in the region of constant thickness, and then

$$\frac{Q}{2\pi} = \frac{UR^2}{2} \left(\frac{\xi_0^2}{R^2} - 1 \right) - \frac{\rho g R^4}{4\mu} \left(\frac{\xi_0^4}{R^4} \ln \frac{\xi_0}{R} - \frac{3\xi_0^4}{4R^4} + \frac{\xi_0^2}{R^2} - \frac{1}{4} \right). \tag{2.14}$$

We derive the position of the stagnation line ξ_s. By definition, at that line the surface velocity of the film $u|_{r=\xi}$ is zero, which, from (2.12), gives

$$\frac{AR^2}{4\mu} = U \left(2 \frac{\xi_s^2}{R^2} \ln \frac{\xi_s}{R} - \frac{\xi_s^2}{R^2} + 1 \right)^{-1},$$

and from (2.13),

$$\frac{Q}{2\pi} = \frac{UR^2}{2} \left(\frac{\xi_s^2}{R^2} - 1 \right) - \frac{UR^4}{2\xi_s^2 \ln \frac{\xi_s}{R} - \xi_s^2 + R^2} \left(\frac{\xi_s^4}{R^4} \ln \frac{\xi_s}{R} - \frac{3\xi_s^4}{4R^4} + \frac{\xi_s^2}{R^2} - \frac{1}{4} \right). \tag{2.15}$$

Substitution of (2.14) into (2.13) and conversion to dimensionless variables and parameters

$$H = \xi/R,\ x = z/R,\ H_s = \xi_s/R,\ S = \xi_0/R,\ \mathrm{Ca} = \mu U/\sigma, \quad \mathrm{Go} = R(\rho g/2\sigma)^{1/2} \tag{2.16}$$

results in an ordinary nonlinear differential equation for the film thickness in the extraction region:

$$\frac{d^3 H}{dx^3} = -\frac{1}{H^2} \frac{dH}{dx} + \frac{8(\mathrm{Ca}\, S^2 - H^2) - 2\mathrm{Go}^2 (4S^2 \ln S - 3S^4 + 4S^2 - 1)}{4H^4 \ln H - 3H^4 + 4H^2 - 1} + 2\mathrm{Go}^2. \tag{2.17}$$

This is true in the range in H from S to H_s, which is determined from (2.15) in dimensionless form:

$$H_s^2 - \frac{4H_s^4 \ln H_s - 3H_s^4 + 4H_s^2 - 1}{2(2H_s^2 \ln H_s - H_s^2 + 1)} = S^2 - \frac{\mathrm{Go}^2}{4\,\mathrm{Ca}} (4S^4 \ln S - 3S^4 + 4S^2 - 1). \tag{2.18}$$

Three arbitrary constants appear on integrating (2.17), which can be found from the conditions

$$H \to S,\ \frac{dH}{dx} \to 0,\ \frac{d^2 H}{dx^2} \to 0 \text{ for } x \to \infty.$$

Equation (2.17) contains also the unknown quantity S or ξ_0 in dimensional form, which is related by (2.14) to Q. To determine this we consider the flow region below the stagnation line.

3. Here the characteristic speed of the liquid is less than the plate extraction rate, while the characteristic scale of the liquid motion is much greater than the thickness of the extracted film for the case of a wide and deep bath. Therefore, the spatial derivatives of

the velocities will be much less than in the extraction region, and they can be neglected in (2.1)-(2.4). Then the surface shape is defined by the Euler relations

$$\partial p/\partial y + \rho g = 0,\ \partial p/\partial r = 0 \tag{3.1}$$

with the boundary condition

$$p_0 - p = \frac{\sigma}{\left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{\frac{1}{2}}}\left\{\frac{d^2\xi}{dy^2}\left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{-1} - \frac{1}{\xi}\right\} \quad \text{at} \quad r = \xi. \tag{3.2}$$

Here the y coordinate coincides with the z axis but is reckoned from the horizontal surface of the liquid in the bath. From (3.1) and (3.2) we get

$$\frac{d^2\xi}{dy^2}\left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{-\frac{3}{2}} - \frac{1}{\xi}\left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{-\frac{1}{2}} = \frac{\rho g}{\sigma} y, \tag{3.3}$$

which coincides with the equation for the shape of the liquid surface under static-meniscus conditions. The solution to (3.3) should be sought with the boundary conditions

$$d\xi/dy = 0 \text{ for } y = b,\ d\xi/dy \to -\infty \text{ for } y \to 0, \tag{3.4}$$

where b is the height to which the liquid rises on the surface of the cylinder under the action of the capillary forces. Numerical integration of the boundary-value problem of (3.3) and (3.4) for an immobile cylinder has given [4] the empirical formula

$$b = \left(\frac{2\rho g}{\sigma}\right)^{\frac{1}{2}} \frac{2{,}4\,\mathrm{Go}^{0,85}}{1+2.4\,\mathrm{Go}^{0,85}},$$

which gives an error of 1% for Go = 3 and which decreases as Go decreases. When the cylinder moves, one has to allow for the effects of the extracted film on the shape of the static meniscus surface. We assume that the static meniscus is attached not to the solid wall but to a liquid film of thickness $h_0 = \xi_0 - R$, which means that in determining the maximum height of rise b one should take ξ_0 instead of R as the radius of the cylindrical body. Then for the static meniscus on extraction

$$b = \left(\frac{2\rho g}{\sigma}\right)^{\frac{1}{2}} \frac{2{,}4\,(\mathrm{Go}\,S)^{0,85}}{1+2{,}4\,(\mathrm{Go}\,S)^{0,85}}. \tag{3.5}$$

Then the problem of (3.3) with (3.4) and (3.5) is converted from a boundary-value problem into a Cauchy one. The solution is readily found by numerical integration.

4. To link up the solutions of (2.17) and (3.3) we specify that the normal stresses acting from the liquid on each of the flow regions are equal on the stagnation line ξ_S, i.e., we equate the right sides of (2.8) and (3.2). As the first term on the right in (2.8) has the same order of smallness along the stagnation line as the terms discarded in (3.2), it should be deleted. Then,

$$\left(\frac{d^2\xi}{dz^2} - \frac{1}{\xi}\right)\Big|_{\xi=\xi_S} = \left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{-\frac{1}{2}}\left\{\frac{d^2\xi}{dy^2}\left[1+\left(\frac{d\xi}{dy}\right)^2\right]^{-1} - \frac{1}{\xi}\right\}\Big|_{\xi=\xi_S},$$

or in dimensionless form on the basis of (2.16) and (3.3)

$$\left(\frac{d^2H}{dx^2} - \frac{1}{H}\right)\Big|_{H=H_S} = 2\,\mathrm{Go}^2 x_1 |_{H=H_S}, \tag{4.1}$$

where $x_1 = y/R$.

Then the scheme for solving the problem is as follows. We specify Ca and Go and choose a certain value for S, and use Newton's method with (2.18) to find H_S. Then we solve (2.17) and (3.3) numerically by the Runge–Kutta method and determine $(d^2H/dx^2 - 1/H)|_{H=H_S}$ from (2.17) and the value of x at which $H = H_S$ from (3.3), and then check for obedience to (4.1). Simple iteration gives the value of S obeying this requirement with given Ca and Go. Figure 2 shows the results, where the ordinate is the dimensionless film thickness

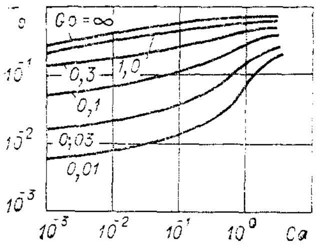

Fig. 2

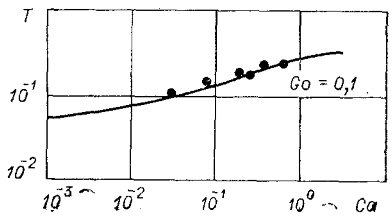

Fig. 3

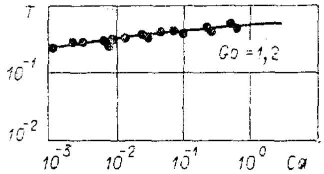

Fig. 4

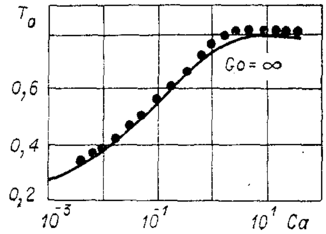

Fig. 5

$$T_0 = (\xi_0 - R)(\rho g/\mu U)^{1/2},$$

which is related to the dimensionless film radius S by

$$T_0 = (S - 1)(2/\mathrm{Ca})^{1/2}\mathrm{Go}.$$

In experiments, one often measures not the film thickness $h_0 = \xi_0 - R$, but the liquid flow rate Q, and the thickness is taken as flow thickness h_∞

$$\pi(h_\infty + R)^2 - \pi R^2 = Q/U.$$

Then from (2.14) and (1.16),

$$\left(\frac{h_\infty}{R} + 1\right)^2 = S^2 - \frac{\mathrm{Go}^2}{\mathrm{Ca}}\left(S^4 \ln S - \frac{3}{4} S^4 + S^2 - \frac{1}{4}\right).$$

We introduce the dimensionless flow thickness $T = h_\infty(\rho g/\mu U)^{1/2}$, so

$$[T\,\mathrm{Go}\,(2/\mathrm{Ca})^{1/2} + 1]^2 = S^2 - \frac{\mathrm{Go}^2}{\mathrm{Ca}}\left(S^4 \ln S - \frac{3}{4} S^4 + S^2 - \frac{1}{4}\right).$$

Figures 3-5 compare the theoretical and experimental results taken from [3, 5], which show good agreement throughout the ranges of extraction speeds used in the experiments.

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